Lecture 8.

As previously advertised, the advantage of the notion of Ψcvx-ity is that it can be checked locally near each boundary point. This is a consequence of:

Thm 3. Let $\Omega \subseteq \mathbb{C}^n$. TFAE:

(i) Ω is Ψcvx, i.e. $-\log d(z, \Omega^c)$ is $PSH(\Omega)$.

(ii) $u(z) = -\log d(z, \Omega^c)$ is PSH near the boundary, i.e., $\exists$ closed $F \subseteq \Omega$ s.t. $u \in PSH(\Omega \setminus F)$.

(iii) $\exists$ cont., PSH exhaustion function v, i.e., $\overline{\{z : v(z) < c\}} \subset\subset \Omega$ for all $c \in \mathbb{R}$.

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are easy. The implication (iii) $\Rightarrow$ (i) goes via an additional (iv) similar to the characterization of DoH by holomorphic hulls of compacts. Details are in Hormander, Sec. 2.6.

Using (ii), one easily sees:

Cor 1. $\Omega \subseteq \mathbb{C}^n$ is Ψcvx $\Leftrightarrow$ $\forall p \in \partial\Omega$ $\exists$ open nbhd U_p of p s.t. $\Omega \cap U_p$ is Ψcvx.

The Levi form and pcvxity

Let us now assume that $M = \partial\Omega \subseteq \mathbb{C}^n$ is a $\mathcal{C}^2$-smooth $(2n-1)$-dim $(\mathbb{R})$ mfld.

Let $\rho = 0$ be a <u>defining function</u> for M, i.e. $\rho : \mathbb{C}^n \to \mathbb{R}$ is $\mathcal{C}^2$, $d\rho \neq 0$ on M, and $M = \{z : \rho(z, \bar{z}) = 0\}$

convenient notion to remind you that when taking derivatives you need <u>both</u> $\frac{\partial \rho}{\partial z_k}$, $\frac{\partial \rho}{\partial \bar{z}_\ell}$, etc., to capture all real derivatives $\frac{\partial \rho}{\partial x_k}, \frac{\partial \rho}{\partial y_\ell}$.

<u>Note</u>. Any two defining functions $\rho, \tilde{\rho}$ are related by $\rho = a\tilde{\rho}$, where $a \in \mathcal{C}^2$ and $a \neq 0$ on M.

<u>Def</u>. For $p \in M$, $T_p^{1,0} M \subseteq T_p^{1,0} \mathbb{C}^n$ consists of $\xi = \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial z_j} \in T_p^{1,0} \mathbb{C}^n$ s.t.

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p)\, \xi_j = 0.$$

It follows easily from the Note above that T_pM is independent of choice of ρ. Moreover, since $\partial\rho = \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} dz_j$ does not vanish on M, T_pM is a $(n-1)$-$\dim_{\mathbb{C}}$ subspace of $T_p\mathbb{C}^n \cong \mathbb{C}^n$.

Thm 1 Let $\Omega \subseteq \mathbb{C}^n$ and assume $M = \partial\Omega$ is C^2 w/ a defining function ρ s.t. $\Omega = \{\rho < 0\}$. Then, Ω is ψcvx $\Leftrightarrow$ $\forall p \in \partial\Omega$

$$(*) \quad \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(p)\, \xi_i \bar{\xi}_j \geq 0, \ \forall \xi \in T_p^{1,0} M$$

Pf. We note that $(*)$ is independent of the choice of defining function. If $\tilde{\rho} = a\rho$ is another (with $a > 0$), then

$$\frac{\partial}{\partial z_i}\tilde{\rho} = a\frac{\partial \rho}{\partial z_i} + \rho\frac{\partial a}{\partial z_i}, \quad \frac{\partial^2 \tilde{\rho}}{\partial z_i \partial \bar{z}_j} = a\frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}$$

$$+ \frac{\partial \rho}{\partial z_i}\frac{\partial a}{\partial \bar{z}_j} + \frac{\partial \rho}{\partial \bar{z}_j}\frac{\partial a}{\partial z_i} + \rho\frac{\partial^2 a}{\partial z_i \partial \bar{z}_j}.$$

Thus, if $p \in M$ ($\Rightarrow \rho(p) = 0$) and $\xi \in T_pM$ ($\Rightarrow \sum_{i=1}^{n} \frac{\partial \rho}{\partial z_i}\xi_i = \sum_{j=1}^{n} \frac{\partial \rho}{\partial \bar{z}_j}\bar{\xi}_j = 0$), we find